11

"ON THE THEORY OF A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE ELLIPTIC-PERABOLIC TYPE"*

By M. RAZIUDDIN SIDDIQI
(Hyderabad—Deccan)

(Received for publication, March 25, 1938.)

ABSTRACT. The "Fourier Method" developed by the author for the unique solution of general non-linear equations of the parabolic and hyperbolic types has been extended to the non-linear partial differential equation of the elliptic-parabolic type

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial t}=u^2$$

and for certain boundary and initial conditions, unique solution has been obtained.

INTRODUCTION

Apart from the intrinsic interest of the subject, the occurrence of non-linear differential equations in modern physical theories, is making it imperative that such equations should be studied systematically. In several recent papers[1] the author has developed a "Fourier method" for the unique solution of general non-linear equations of the parabolic and hyperbolic types.

In this paper the method is extended to the non-linear partial differential equation of the elliptic-parabolic type

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=u^2$$

for the boundary conditions:

$$u\,(o,\,y;\,t)=u\,(\pi,\,y;\,t)=o;\quad u\,(x,\,o,\,t)=u(x,\,\pi,\,t)=o,$$

and the initial conditions

$$u\,(x,\,y;\,o)=f\,(x,\,y)=\sum_{m,n} C_{m,n}\sin m\,x\sin ny.$$

* Communicated by the Indian Physical Society.

If it is assumed that $\sum\limits_{m,n} (m^2+n^2) \; | C_{m,n} | = C$, then it is proved that one and only one regular solution $u(x, y; t)$ exists, which can be expressed in the form of a double Fourier series :

$$u(x, y; t) = \sum_{m,n} v_{m,n}(t) \sin mx \sin ny.$$

The coefficients $v_{m,n}(t)$ are determined with the help of a doubly infinite system of non-linear integral equations which is solved by the method of successive approximations.

It is found that there is no restriction on the value of t, which can extend to infinity. On the other hand, the given function $f(x, y)$ must be of a more restricted character, *viz.*, such that C must not exceed a certain constant. This is in general the case for parabolic equations. For hyperbolic equations, the function f is more general, but the domain in t is more restricted.

EXISTENCE OF THE SOLUTION

We consider the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = u^2 \qquad \dots (1)$$

and determine a solution $u(x, y, t)$ which is regular in the domain R defined by :

$$0 \leq x \leq \pi, \; 0 \leq y \leq \pi, \; 0 \leq t, \qquad \dots (2)$$

and which satisfies the boundary conditions :

$$u(o, y; t) = u(\pi, y; t) = 0 \text{ for all } y \text{ and } t \text{ in R}, \qquad \dots (3)$$

$$u(x, o; t) = u(x, \pi; t) = 0 \text{ for all } x \text{ and } t \text{ in R},$$

and

$$u(x, y; o) = f(x, y) \text{ for all } x \text{ and } y \text{ in R}. \qquad \dots (4)$$

We assume that $f(x, y)$ can be expanded in the double Fourier series :*

$$f(x, y) = \sum_{m,n}{}^{*} C_{m,n} \sin mx \sin ny, \qquad \dots (5)$$

such that the series $\sum\limits_{m,n} (m^2+n^2) \; | C_{m,n} |$ is convergent.

* Unless otherwise stated, summation is to be extended from 1 to ∞ throughout this paper.

For the solution of (1) we write

$$u(x, y; t) = \sum_{m,n} v_{m,n}(t) \sin mx \sin ny, \qquad \dots \quad (6)$$

and get

$$u^2(x, y, t) = \sum_{m,n} \left\{ \frac{4}{\pi^2} \int_0^\pi \int_0^\pi u^2(\alpha, \beta; t) \sin m\alpha \sin n\beta \, d\alpha \, d\beta \right\} \sin mx \sin ny,$$

$$= \sum_{m,n} Z_{m,n}(t) \sin mx \sin ny, \qquad \dots \quad (7)$$

where

$$Z_{m,n}(t) = \sum_{\substack{\kappa,\lambda,\\ \mu,\nu}} f_m(\kappa, \lambda) f_n(\mu, \nu) v_{\kappa,\mu}(t) v_{\lambda,\nu}(t), \qquad \dots \quad (8)$$

with

$$\left. \begin{aligned} f_m(k, \lambda) &= \frac{2}{\pi} \int_0^\pi \sin kx \sin \lambda x \sin mx \, dx, \\ f_n(\mu, \nu) &= \frac{2}{\pi} \int_0^\pi \sin \mu y \sin \nu y \sin ny \, dy, \end{aligned} \right\} \qquad \dots \quad (9)$$

From (1), (6) and (7) we get then

$$-\sum_{m,n} (m^2 + n^2) v_{m,n}(t) \sin mx \sin ny - \sum_{m,n} \frac{dv_{m,n}}{dt} \sin mx \sin ny$$

$$= \sum_{m,n} Z_{m,n}(t) \sin mx \sin ny ; \qquad \dots \quad (10)$$

therefore for all $m, n \geq 1$.

$$\frac{dv_{m,n}}{dt} + (m^2 + n^2) v_{m,n}(t) = -Z_{m,n}(t). \qquad \dots \quad (11)$$

We have assumed here, what we shall prove later, that all the series in (10) are absolutely and uniformly convergent in R.

From (4) and (5) we see that the solution (6) would have to satisfy the initial condition:

$$v_{m,n}(0) = C_{m,n} \quad (m, n = 1, 2, \dots) \qquad \dots \quad (12)$$

The solution of (11) which satisfies the initial condition (12) is given by

$$v_{m,n}(t) = C_{m,n} e^{-(m^2+n^2)t} - \int_0^t e^{-(m^2+n^2)(t-s)} Z_{m,n}(s) \, ds$$

$$=C_{m,n}e^{-(m^2+n^2)t}-\int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\times \sum_{k\lambda\mu\nu} f_m(k,\lambda)\, f_n(\mu,\nu) v_{k,\mu}(s)\; v_{\lambda,\nu}(s)\, ds. \quad . \quad (13)$$

We set for all $m,\ n \geq 1$:

$$w_{m,n}(t)=(m^2+n^2)\; v_{m,n}(t),\;\; C'_{m,n}=(m^2+n^2)\; C_{m,n} \qquad \ldots \quad (14)$$

and get from (13)

$$w_{m,n}(t)=C'_{m,n}\; e^{-(m^2+n^2)t} - (m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\times \sum_{k\lambda\mu\nu} \frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)}\; w_{k,\mu}(s)\; w_{\lambda,\nu}(s)\, ds,$$

$$(m,n=1,\ 2,\ldots\ldots). \qquad \ldots \quad (15)$$

This is a doubly infinite system of non-linear integral equations for the determination of the functions $w_{m,n}(t)$ which in their turn determine the Fourier co-efficient $v_{m,n}(t)$.

We shall solve this system by the method of successive approximations, and for this purpose, we set

$$\overset{(o)}{w}_{m,n}(t)=C'_{m,n}\; e^{-(m^2+n^2)t} \qquad \ldots \quad (16)$$

and for all $r \geq 1$:

$$\overset{(r)}{w}_{m,n}(t)=C'_{m,n}\; e^{-(m^2+n^2)t} - (m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\times \sum_{k\lambda\mu\nu} \frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)}\; \overset{(r-1)}{w}_{k,\mu}(s)\; \overset{(r-1)}{w}_{\lambda,\nu}(s)\, ds \qquad \ldots \quad (17)$$

We have to prove the convergence of this approximation

We shall first show that the doubly infinite series

$$\sum_{m,n} \frac{|\, f_m(k,\lambda)\, f_n(\mu,\nu)\, |}{(k^2+\mu^2)(\lambda^2+\nu^2)} \qquad \ldots \quad (18)$$

is uniformly convergent for all $k,\ \mu,\ \lambda,\ \nu$.

We have on writing $\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$

$$\sin mx \sin ny = -\frac{1}{(m^2+n^2)} \Delta (\sin mx \sin ny).$$

Therefore

$$f_m(k,\lambda)\, f_n(\mu,\nu) = -\frac{4}{(m^2+n^2)\pi^2} \int_0^\pi \int_0^\pi \sin kx \sin \mu y \sin \lambda x \sin \nu y \times \Delta (\sin mx \sin ny)\, dx\, dy$$

$$= -\frac{4}{(m^2+n^2)\pi^2} \int_0^\pi \int_0^\pi \sin mx \sin ny \times \Delta (\sin kx \sin \mu y \sin \lambda x \sin \nu y)\, dx\, dy$$

$$= -\frac{4}{(m^2+n^2)\pi^2} \int_0^\pi \int_0^\pi \sin mx \sin ny \Big\{ -(k^2+\lambda^2) \sin kx \sin \mu y \times \sin \lambda x \sin \nu y$$
$$+ 2k\lambda \cos kx \cos \lambda x \sin \mu y \sin \nu y - (\mu^2+\nu^2) \sin kx \sin \mu y \sin \lambda x \sin \nu y$$
$$+ 2\mu\nu \sin kx \sin \lambda x \cos \mu y \cos \nu y \Big\} dx\, dy.$$

Therefore

$$\frac{|f_m(k,\lambda)\, f_n(\mu,\nu)|}{(k^2+\mu^2)(\lambda^2+\nu^2)} < \frac{4}{(m^2+n^2)} \Big\{ |f_m(k,\lambda)\, f_n(\mu,\nu)| + |g_m(k,\lambda)\, f_n(\mu,\nu)| + |f_m(k,\lambda)\, g_n(\mu,\nu)| \Big\} \quad \ldots (19)$$

where

$$\left.\begin{aligned} g_m(k,\lambda) &= \frac{2}{\pi}\int_0^\pi \cos kx \cos \lambda x \sin mx\, dx, \\ g_n(\mu,\nu) &= \frac{2}{\pi}\int_0^\pi \cos \mu x \cos \nu x \sin nx\, dx. \end{aligned}\right\} \quad \ldots (20)$$

From Parsevals' theorem we know that for all $k,\lambda,\ \mu,\ \nu$:

$$\sum_m f_m^2\,(k,\lambda) \leq 2,\ \sum_n f_n^2(\mu,\nu) \leq 2,$$

$$\sum_m g_m^2(k,\lambda) < 2,\ \sum_n g_n^2(\mu,\nu) \leq 2,$$

Therefore

$$\sum_{m,n} f_m^2(k,\lambda)\, f_n^2(\mu,\nu) \leq 4,\ \sum_{m,n} g_m^2(k,\lambda)\, f_n^2(\mu,\nu) \leq 4,$$

$$\sum_{m,n} f_m^2(k,\lambda)\, g_n^2(\mu,\nu) \leq 4.$$

Thus from (19) we get on summing, squaring and using Schwarz's inequality as well as the inequality $2ab \geq a^2 + b^2$:

$$\left\{\sum_{m,n} \frac{|\, f_m(k,\lambda)\, f_n(\mu,\nu)\,|}{(k^2+\mu^2)(\lambda^2+\nu^2)}\right\}^2 < \left\{\sum_{m,n} \frac{4}{m^2+n^2}\Big[\,|\, f_m(k,\lambda)\, f_n(\mu,\nu)\,| + |\, g_m(k,\lambda)\, f_n(\mu,\nu)\,| + |\, f_m(k,\lambda)\, g_n(\mu,\nu)\,|\,\Big]\right\}^2$$

$$\leq \sum_{m,n} \frac{16}{(m^2+n^2)^2}\cdot \sum_{m,n}\Big\{|\, f_m(k,\lambda)\, f_n(\mu,\nu)\,| + |\, g_m(k,\lambda)\, f_n(\mu,\nu)\,| + f_m\,|\,(k,\lambda)\, g_n(\mu,\nu)\,|\Big\}^2$$

$$\leq \sum_{m,n} \frac{16\times 3}{(m^2+n^2)^2}\cdot \sum_{m,n}\Big\{f_m^2(k,\lambda)\, f_n^2\,(\mu,\nu) + g_m^2(k,\lambda)\, f_n^2(\mu,\nu) + f_m^2(k,\lambda)\, g_n^2\,(\mu,\nu)\Big\}$$

$$< \sum_{m,n} \frac{16\times 3\times 12}{(m^2+n^2)^2}\ .$$

But the doubly infinite series $\sum_{m,n} \frac{1}{(m^2+n^2)^2}$ is convergent, so that $\left\{\sum_{m,n} \frac{|\, f_m(k,\lambda)\, f_n(\mu,\nu)\,|}{(k^2+\mu^2)(\lambda^2+\nu^2)}\right\}^2$ and with it the series (18) is uniformly convergent for all $k,\ \lambda,\ \mu,\ \nu$:

$$\sum_{m,n} \frac{|\, f_m\,(k,\,\lambda)\; f_n\,(\mu,\,\nu)|}{(k^2+\mu^2)\,(\lambda^2+\nu^2)} = a \text{ (say)} \qquad \ldots \quad (21)$$

where is an absolute constant.

We write, since $\sum_{m,n}(m^2+n^2)\,C_{m,n}$ is supposed to be convergent,

$$\sum_{m,n}|C'_{m,n}|=\sum_{m,n}(m^2+n^2)\,|C_{m,n}|=C. \qquad (22)$$

We have also for all $t>0$ and all $m,n\geq 1$.

$$\left|\int_0^t e^{-(m^2+n^2)(t-s)}\,ds\right|=\left|e^{-(m^2+n^2)t}\int_0^t e^{(m^2+n^2)s}\,ds\right|$$

$$=\left|\frac{1}{e^{(m^2+n^2)t}}\cdot\frac{e^{(m^2+n^2)t}-1}{m^2+n^2}\right|<\frac{1}{(m^2+n^2)} \qquad \ldots\ (23)$$

Thus from (21) and (23) we get for all $t>0$ and all $k,\lambda,\mu,\nu\geq 1$.

$$\sum_{m,n}(m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}\,ds\,.\,\frac{|f_m(k,\lambda)f_n(\mu,\nu)|}{(k^2+\mu^2)(\gamma^2+\nu^2)}$$

$$<\sum_{m,n}(m^2+n^2).\frac{1}{(m^2+n^2)}\cdot\frac{|f_m(k,\lambda)f_n(\mu,\nu)|}{(k^2+\mu^2)(\lambda^2+\nu^2)}$$

$$<a\,. \qquad \ldots\ (24)$$

Thus if $\sum_{m,n}|w^{(r-1)}_{m,n}(t)|$ converges uniformly for all t, we get from (17) and (22):

$$\sum_{m,n}\left|w^{(r)}_{m,n}(t)\right|<C+a\left\{\mathrm{Max}\sum_{m,n}|w^{(r-1)}_{m,n}(t)|\right\}^2 \qquad \ldots\ (25)$$

From (16) we get for all t

$$\sum_{m,n}|w^{(0)}_{m,n}(t)|\times\leq\sum_{m,n}|C'_{m,n}|=C. \qquad \ldots\ (26)$$

Substituting this in (25) for $r=1$.

$$\sum_{m,n}|w^{(1)}_{m,n}(t)|>C+aC^2.$$

We assume that

$$C<\frac{1}{4a}\,, \qquad \ldots\ (27)$$

so that we get

$$\sum_{m,n}\left|w^{(1)}_{m,n}(t)\right|<C+C=2C.$$

Substituting this again in (25) for $r=2$, we get for all t :

$$\sum_{m,n}\left|w^{(2)}_{m,n}(t)\right| < C + a(2C)^2$$
$$< 2C.$$

In general, for all $r \geq 1$, and for all $t > 0$, we get

$$\sum_{m,n}\left|w^{(r)}_{m,n}(t)\right| < 2C. \qquad \dots (28)$$

We shall prove further that the triple series

$$\sum_{r=0}^{\infty}\sum_{m,n}\left|w^{(r+1)}_{m,n}(t) - w^{(r)}_{m,n}(t)\right| \qquad \dots (29)$$

converges uniformly for all t.

We have

$$w^{(r+1)}_{m,n}(t) - w^{(r)}_{m,n}(t) = -(m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$
$$\times \sum_{k\lambda\mu\nu}\frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)}\left\{ w^{(r)}_{k,\mu}(s)\quad w^{(r)}_{\lambda,\nu}(s) - w^{(r-1)}_{k,\mu}(s) - w^{(r-1)}_{\lambda,\nu}(s)\right\} ds,$$
$$= -(m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}\sum_{k\lambda\mu\nu}\frac{f_m(k,\gamma)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)}$$
$$\times\left\{ w^{(r)}_{k,\mu}(s)\left[w^{(r)}_{\lambda,\nu}(s) - w^{(r-1)}_{\lambda,\nu}(s)\right] + w^{(r-1)}_{\lambda,\nu}(s)\left[w^{(r)}_{k,\mu}(s) - w^{(r-1)}_{k,\mu}(s)\right]\right\} ds.$$

Therefore summing over m, n and taking account of (21) and (29) we get for all $t \geq 0$:

$$\sum_{m,n}\left|w^{(r+1)}_{m,n}(t) - w^{(r)}_{m,n}(t)\right| < a.\,2.\,2C\ \mathrm{Max}\sum_{m,n}\left|w^{(r)}_{m,n}(t) - w^{(r-1)}_{m,n}(t)\right| \qquad \dots (30)$$

Repeating this reduction r times, we get for all $t > 0$:

$$\sum_{m,n}\left|w^{(r+1)}_{m,n}(t) - w^{(r)}_{m,n}(t)\right| < (4aC)^r\ \mathrm{Max}\sum_{m,n}\left|w^{(1)}_{m,n}(t) - w^{(0)}_{m,n}(t)\right| \qquad \dots (31)$$

Now,

$$w^{(1)}_{m,n}(t) - w^{(0)}_{m,n}(t) = -(m^2+n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$
$$\sum_{k\lambda\mu\nu}\frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)} \times w^{(0)}_{k\mu}(s)\, w^{(0)}_{\lambda\nu}(s)\, ds$$

and $\sum_{m,n} \left| w^{(o)}_{m,n}(t) \right| \leq C$ for all t as shown in (26).

Therefore for all $t > o$

$$\sum_{m,n} \left| w^{(1)}_{m,n}(t) - w^{(o)}_{m,n}(t) \right| < aC^2.$$

Consequently, we have from (31) on summing over r from o to ∞, and remarking that in virtue of (27) $4\,aC < 1$:

$$\sum_{r=o}^{\infty} \sum_{m,n} \left| w^{(r+1)}_{m,n}(t) - w^{(r)}_{m,n}(t) \right| < aC^2 \sum_{r=o}^{\infty} (4aC)^r$$

$$< \frac{aC^2}{1 - 4aC}. \qquad \ldots \quad (32)$$

This shows that the series (29) is uniformly convergent for all t.

From this we conclude that all the limits

$$w_{m,n}(t) = \lim_{r \to \infty} w^{(r)}_{m,n}(t) \qquad (m, n = 1, 2 \ldots..) \qquad \ldots \quad (33)$$

exist, and that $w_{m,n}(t)$ is continuous. Moreover, for all t :

$$\sum_{m,n} \left| w_{m,n}(t) \right| < 2C. \qquad \ldots \quad (34)$$

From (17) on proceeding to $r \to \infty$, we get

$$w_{m,n}(t) = C'_{m,n}\, e^{-(m^2+n^2)\,t} - (m^2+n^2) \int_o^t e^{-(m^2+n^2)\,(t-s)}.$$

$$\sum_{k\lambda\mu\nu} \frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda+\nu^2)}\; w_{k,\mu}(s)\; w_{\lambda,\nu}(s)\; ds. \qquad \ldots \quad (35)$$

Writing again

$$v_{m,n}(t) = \frac{1}{(m^2+n^2)}\, w_{m,n}(t) \quad ,$$

$$C_{m,n} = \frac{1}{(m^2+n^2)}\, C'_{m,n}, \qquad \ldots \quad (36)$$

we obtain

$$v_{m,n}(t) = C_{m,n}\, e^{-(m^2+n^2)t}$$

$$- \int_o^t e^{-(m^2+n^2)\,(t-s)} \sum_{k\lambda\mu\nu} f_m(k,\lambda)\, f_n(\mu,\nu)\,.\quad v_{k,\mu}(s)\; v_{\lambda,\nu}(s) ds \qquad \ldots \quad (37)$$

Thus for all $m, n \geq 1$, $v_{m,n}(t)$ satisfies the integral equation (13) and consequently the differential equation (11).

We have proved in (34) that $\sum_{m,n} |w_{m,n}(t)| = \sum_{m,n} (m^2+n^2) \, | v_{m,n}(t) |$ is uniformly convergent; we also find that $\sum_{m,n} | Z_{m,n}(t) |$ is also uniformly convergent. We conclude from (11) that $\sum_{m,n} \left| \frac{d v_{m,n}}{dt} \right|$ is uniformly convergent. Thus we have shown that all the series in (10) are absolutely and uniformly convergent in R.

The function

$$u(x, y\,;\, t) = \sum_{m,n} v_{m,n}(t) \sin mx \sin ny \qquad \ldots \quad (38)$$

is therefore the required solution of the equation (1) which satisfies the conditions (3) and (4).

We have to prove now that this is the only solution of its kind. Obviously, it is enough for this purpose to show that the system of integral equation (15) has no other solution than (33), which is such that (34) is also satisfied.

If possible, suppose that (15) has another solution $\bar{w}_{m,n}(t)$ $(m, n = 1, 2, \ldots)$ and that $\sum_{m,n} | \bar{w}_{m,n}(t) | < 2C.$ $\qquad \ldots \quad (39)$

Then we have:

$$\bar{w}_{m,n}(t) - \overset{(r)}{w}_{m,n}(t) = -(m^2+n^2) \int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\sum_{k\lambda\mu\nu} \frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)} \left\{ \bar{w}_{k,\mu}(s)\, \bar{w}_{\lambda,\nu}(s) - \overset{(r-1)}{w}_{k,\mu}(s)\, \overset{(r-1)}{w}_{\lambda,\nu}(s) \right\} ds$$

$$= -(m^2+n^2) \int_0^t e^{-(m^2+n^2)(t-s)} \sum_{k,\lambda,\mu,\nu} \frac{f_m(k,\lambda)\, f_n(\mu,\nu)}{(k^2+\mu^2)(\lambda^2+\nu^2)} .$$

$$\left\{ \bar{w}_{k,\mu}(s) \left[\bar{w}_{\lambda,\nu}(s) - \overset{(r-1)}{w}_{k,\nu}(s) \right] + \overset{(r-1)}{w}_{\lambda,\nu}(s) \left[\bar{w}_{k,\mu}(s) - \overset{(v-1)}{w}_{k,\mu}(s) \right] \right\} ds.$$

Then on account of (21), (28) and (39) we have for all t:

$$\sum_{m,n} \left| \bar{w}_{m,n}(t) - \overset{(r)}{w}_{m,n}(t) \right| < (a.\ 2.\ 2C.\ \text{Max} \sum_{m,n} \left| \bar{w}_{m,n}(t) - \overset{(r-1)}{w}_{m,n}(t) \right|. \qquad \ldots \quad (40)$$

Repeating this reduction r-times we get

$$\sum_{m,n} \left| \bar{w}_{m,n}(t) - \overset{(r)}{w}_{m,n}(t) \right| < (4\, aC)^r \ \text{Max} \sum_{m,n} \left| \bar{w}_{m,n}(t) - \overset{(0)}{w}_{m,n}(t) \right| . \qquad \ldots \quad (41)$$